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A deformation of quantum mechanics

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Abstract. On a one-dimensional lattice without uniform interval, the Hermitian conjugation of a q-differential operator is discussed. Then a deformation of quantum mechanics in one dimension is presented. As an application, the harmonic oscillator is discussed. The energy spectrum and the eigenfunctions are shown to depend on an arbitrary deformation function. The deformed coherent states are also discussed. It is found that the completeness relation of coherent states holds for the case of q-coherent states, i.e. the deformation of the Heisenberg–Weyl algebra is a q-analogue Hopf algebra.

1. Introduction

Recently, much attention has been paid to the study of so-called quantum groups in both physical and mathematical aspects. Quantum groups [1] are the dual category of Hopf algebra which are neither commutative nor co-commutative. Most of the well studied concrete examples of quantum groups are deformations of the universal enveloping algebra of the semi-simple Lie algebra [2–5]. These mathematical and algebraic structures arise in quantum inverse scattering theory [6] and statistical mechanics [7]. They may be thought of as matrix groups in which the elements themselves are not commutative but obey a set of bilinear product relations [4, 8].

Dozens of works are devoted to the study of $U_q(\mathrm{Sl}(2))$ etc [9] via the so-called q-deformation of bosonic realization, which is a q-analogue of the Schwinger technique in quantum mechanics [10]. It is known, by the work of Manin [11] and Woronowicz [12] and further development by Wess and Zumino [13], that quantum groups provide a concrete example of non-commutative differential geometry. A connection between quantum group and Lie-admissible Q-algebra was realized in the work of Janussis [14].

As one of the attempts to explore the physical significance of quantum groups, a q-extension of a one-dimensional harmonic oscillator in the Schrödinger picture is given in the work of Minahan [15]. We try to establish a deformation of quantum mechanics so that standard quantum mechanics is its limit case, and quantum group symmetry is contained in it. To do so is not only to explore the meaning of quantum groups in the context of quantum mechanics, but to provide possibilities of non-perturbation explanations of some perturbation corrections [16] as well. The present paper reports one step toward the above goal. In the next section, we briefly illustrate some notations and derive some useful formulae of q-differential calculus. In section 3, we discuss Hermitian conjugation and establish a deformation of the one-dimensional stationary Schrödinger equation. In section 4, we discuss harmonic oscillators. The energy spectrum and eigenfunctions are found to depend on a function, which involves concrete deformations. In section 5, we discuss deformations of coherent states, especially q-coherent states and the q-coherent states representation, an analogue of Bargman space representation. Finally, we give some conclusions and a discussion.

2. The q-differential integral calculus

The lattice formulation of quantum field theories allows the non-perturbation calculation of bound-state mass and decay amplitude. In the usual lattice approach to the Schrödinger equation either using finite-difference [17] or using finite-element [18] methods, the lattice spacing is the same everywhere, i.e. the coordinates of lattice points are an arithmetic sequence. However, it is known that eigenfunctions of bound states always descend properly as the coordinate goes to infinity such that they are square-integrable, i.e. L^2-functions in mathematical terminology. By means of the basic idea of the Lebesque integral, we may no longer consider a lattice with a constant lattice spacing. The lattice spacing should increase as the coordinates of lattice points do. Evidently, the simplest case is one in which the coordinates of lattice points are in a geometric sequence. This can be regarded as a lattice deformation from an arithmetic lattice to a geometric one (q-lattice) i.e.

$$\{x_n = na + x_0 \mid n \in Z\} \to \{x'_n = q^{2n} x'_0 \mid n \in Z\} \tag{2.1}$$

which is realized by exponential map $x'_n = \mathrm{e}^{x_n}$ $(q := \mathrm{e}^{a/2})$.

On the basis of the above consideration, we will re-introduce the following definition and formulations. For $x \in \mathcal{R}$, let $\mathcal{F}$ denotes the set of all complex functions on $\mathcal{R}$, i.e. the composition of

$$\mathcal{F} = \{f \mid f(x) \in \mathcal{C}\} = \mathrm{Fun}(\mathcal{R}, \mathcal{C})$$

is defined by

$$(f \circ g)(x) = f(x)g(x) \qquad \forall f, g \in \mathcal{F}.$$

Now we introduce a dilation operator $\hat{q}$

$$\hat{q} : \mathcal{F} \to \mathcal{F}$$

defined by

$$(\hat{q}f)(x) = f(qx) \qquad f \in \mathcal{F} \tag{2.2}$$

(obviously $(\hat{q})^{-1} = (\hat{q}^{-1})$). Then a q-difference can be defined by

$$\mathrm{d}_q f := \hat{q} f - \hat{q}^{-1} f \,. \tag{2.3}$$

The q-difference quotient operator is given by

$$\frac{\mathrm{d}_q}{\mathrm{d}_q \mathrm{id}_{\mathcal{R}}} = \frac{\hat{q} - \hat{q}^{-1}}{(q - q^{-1})\mathrm{id}_{\mathcal{R}}} \tag{2.4a}$$

or explicitly

$$\frac{\mathrm{d}_q f(x)}{\mathrm{d}_q x} = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x} \tag{2.4b}$$

which is invariant under $q \to q^{-1}$, and which recovers the usual definition of function derivation when $q \to 1$. The q-analogue of Lebnitz rule is easily obtained from definition (2.4):

$$\frac{\mathrm{d}_q}{\mathrm{d}_q x}(f \circ g) = \frac{\mathrm{d}_q f}{\mathrm{d}_q x} \circ \hat{q} g + (\hat{q}^{-1} f) \circ \frac{\mathrm{d}_q g}{\mathrm{d}_q x} = \frac{\mathrm{d}_q f}{\mathrm{d}_q x} \circ \hat{q}^{-1} g + (\hat{q} f) \circ \frac{\mathrm{d}_q g}{\mathrm{d}_q x} . \tag{2.5}$$

If we let $f(x) = x$, (2.5) gives an operator relation

$$\frac{\mathrm{d}_q}{\mathrm{d}_q x} x - q^{-1} x \frac{\mathrm{d}_q}{\mathrm{d}_q x} = \hat{q} . \tag{2.6}$$

Similarly we have another useful operator relation

$$\hat{q} \frac{\mathrm{d}_q}{\mathrm{d}_q x} - q^{-1} \frac{\mathrm{d}_q}{\mathrm{d}_q x} \hat{q} = 0 . \tag{2.7}$$

One may define a q-integral as the inverse of the q-difference quotient, denoted by

$$\int f(x)\, \mathrm{d}_q x = F(x) + C \tag{2.8}$$

where $\mathrm{d}_q F(x)/\mathrm{d}_q x = f(x)$. Then one can show that the summation along a q-lattice can be calculated from a q-analogue of the Newton–Lebnitz formula

$$\sum_{l=i}^{f} f(q^{2l} x)(q - q^{-1}) q^{2l} x = \int_{x_i}^{x_f} f(x)\, \mathrm{d}_q x = F(x_f) - F(x_i) \tag{2.9}$$

where $x_f = q^f x$, $x_i = q^i x$ (if $x_f x_i < 0$, the summation should be divided into two parts, i.e. x_i to 0 and 0 to x_f). Evidently, the following identity holds:

$$\int_{-\infty}^{+\infty} f(x)\, \mathrm{d}_q x = \int_{-\infty}^{+\infty} q^l \hat{q}^l f(x)\, \mathrm{d}_q x . \tag{2.10}$$

The inverse of the Jackson [19] q-integral (2.4) was first used to study the relation between rational conformal field theories and quantum groups in [20]. In [21] one can find some discussions about q-integration rules.

3. Hermitian conjugation and the q-Schrödinger equation

In this section, we will try to establish the q-Schrödinger equation in coordinate representation. We define the inner product as

$$\langle \psi \mid \varphi \rangle = \int_{-\infty}^{+\infty} \psi(x)^* \varphi(x) \, \mathrm{d}_q x \,. \tag{3.1}$$

We consider the case in which wavefunctions are continuous at the origin and vanish at infinity i.e.

$$\psi(0^+) = \psi(0^-) \qquad \psi(\infty) = 0 \,. \tag{3.2}$$

Using (2.5) and (2.10), we deduce from

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}_q}{\mathrm{d}_q x} (\psi(x)^* \varphi(x)) \, \mathrm{d}_q x = 0$$

that

$$\left[(q\hat{q})^{-1} \frac{\mathrm{d}_q}{\mathrm{d}_q x} \right]^\dagger = -(q\hat{q}) \frac{\mathrm{d}_q}{\mathrm{d}_q x} \,. \tag{3.3}$$

Similarly we have from (2.10) that

$$\hat{q}^\dagger = q^{-1} \hat{q}^{-1} \,. \tag{3.4}$$

Then we obtain from (3.3), (3.4) and (2.6)

$$\left(\frac{\mathrm{d}_q}{\mathrm{d}_q x} \right)^\dagger = - \frac{\mathrm{d}_q}{\mathrm{d}_q x} \,. \tag{3.5}$$

Thus a deformation of a time-independent Schrödinger equation with positive-definite energy spectrum can be defined by the following Hamiltonian:

$$H_q = - \frac{\mathrm{d}_q^2}{\mathrm{d}_q x^2} + V(x) \tag{3.6}$$

where $V(x)$ stands for the potential. Obviously, it reverts to the standard quantum mechanics when $q \to 1$.

4. Harmonic oscillator

Let us consider a harmonic oscillator whose potential is $V(x) = x^2$. Then the Hamiltonian reads

$$H_q = - \frac{\mathrm{d}_q^2}{\mathrm{d}_q x^2} + x^2 \,. \tag{4.1}$$

It can be written as

$$H_q = \tfrac{1}{2}(a_q^+ a_q + a_q a_q^+) \tag{4.2a}$$

where

$$a_q = x + \frac{\mathrm{d}_q}{\mathrm{d}_q x} \qquad a_q^+ = (a_q)^\dagger = x - \frac{\mathrm{d}_q}{\mathrm{d}_q x}\,. \tag{4.2b}$$

It is easy to check that a_q and a_q^+ are no longer energy step decrease and step increase operators, i.e. $[H_q, a_q] \neq -a_q$. However, we can introduce a new Hermitian operator N_q such that

$$[N_q, a_q] = -a_q \qquad [N_q, a_q^+] = a_q^+\,. \tag{4.3}$$

Of course the latter is an immediate consequence of the former in (4.3) due to $N_q^\dagger = N_q$. Obviously $[N_q, H_q] = 0$, so an eigenstate of N_q is also an eigenstate of H_q. If we assume $[a_q, a_q^+] = X$, from Jacobi identity

$$[[N_q, a_q], a_q^+] + [[a_q, a_q^+], N_q] + [[a_q^+, N_q], a_q] = 0$$

we have

$$[[a_q, a_q^+], N_q] = 0\,. \tag{4.4}$$

This shows that $X = \mu(N_q)$, i.e. X may be any function of N_q. Then

$$[a_q, a_q^+] = \mu(N_q)\,. \tag{4.5}$$

In order to recover standard quantum mechanics, the function must go to unity as the deformation parameter q goes to 1. It is known that $\mu(N_q) = [N_q + 1] - [N_q]$ where $[x] := (q^x - q^{-x})/(q - q^{-1})$ is the case indicated by Biedenharn in the study of quantum groups [9]. For a given function μ, chosen in accordance with experiment results, the Hamiltonian (4.2) becomes

$$H_q = a_q^+ a_q + \tfrac{1}{2}\mu(N_q)\,. \tag{4.6}$$

Commutator relations (4.3) and (4.5) are the defining relations of a deformed Heisenberg–Weyl algebra. From those defining relations, one can find that a_q^+ and a_q are creation and annihilation operators of eigenvalues of N_q, a quantum number operator, i.e.

$$N_q|n\rangle = n|n\rangle \qquad a_q^+|n\rangle = \left(\sum_{i=0}^{n}\mu(i)\right)^{1/2}|n+1\rangle \qquad a_q|n\rangle = \left(\sum_{i=0}^{n-1}\mu(i)\right)^{1/2}|n-1\rangle\,. \tag{4.7}$$

Then eigenvalues of Hamiltonian (4.6) are easily calculated:

$$E_q(n) = \sum_{i=0}^{n-1}\mu(i) + \tfrac{1}{2}\mu(n)\,. \tag{4.8}$$

The set of eigenstates $\{|n\rangle \mid n = 0, 1, 2, \ldots, \infty\}$ span a Fock space. In terms of vacuum state $|0\rangle$ (i.e. ground state) the normalized eigenstates in Fock representation are expressed as

$$|n\rangle = \frac{(a_q^+)^n}{[\prod_{j=1}^n (\sum_{i=0}^{j-1} \mu(i))]^{1/2}} \Big|0\Big\rangle . \tag{4.9}$$

Then the eigenfunctions in coordinate representation can be derived from (4.9) without much difficulty. First we consider the vacuum state $|0\rangle$ which satisfies

$$a_q |0\rangle = 0 . \tag{4.10}$$

Using the expression of a_q in coordinate representation (4.2*b*), we have the following q-differential equation:

$$\left(x + \frac{\mathrm{d}_q}{\mathrm{d}_q x} \right) \psi_0 = 0 \tag{4.11}$$

where $\psi_0 := \langle x|0\rangle$. Solving (4.11), we have the eigenfunction of ground state

$$\psi_0(x) = \frac{1}{\pi^{1/4}} \exp_{q^2}(-x^2/[2]) \tag{4.12}$$

where $\exp_q x := \sum_{n=0}^{\infty} \frac{x^n}{[n]!}$. Then we obtain eigenfunctions of excited states

$$\psi_n(x) := \langle x|n\rangle = \frac{1}{\left\{ \sqrt{\pi} \prod_{j=1}^n [\sum_{i=0}^{j-1} \mu(i)] \right\}^{1/2}} \left(x - \frac{\mathrm{d}_q}{\mathrm{d}_q x} \right)^n \exp_{q^2}(-x^2/[2]) . \tag{4.13}$$

5. Coherent states

We now observe the spectrum problem of q-annihilation operator a_q. The eigenstates of a_q

$$a_q |\alpha\rangle = \alpha |\alpha\rangle \tag{5.1}$$

are a deformation of usual coherent states[22]. In Fock representation, (5.1) is easily solved by using (4.7)

$$\begin{aligned} |\alpha\rangle &= [\widetilde{\exp}_\mu(-|\alpha|^2)]^{1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\left\{ \prod_{j=1}^n [\sum_{i=0}^{j-1} \mu(i)] \right\}^{1/2}} \Big| n \Big\rangle \\ &= [\widetilde{\exp}_\mu(-|\alpha|^2)]^{1/2} \, \widetilde{\exp}_\mu(\alpha a_q^+) |0\rangle . \end{aligned} \tag{5.2}$$

Where α takes any value in the complex plane and $\widetilde{\exp}_\mu(x)$ stands for a deformed exponential function

$$\widetilde{\exp}_\mu(x) := \sum_{n=0}^{\infty} \frac{x^n}{\prod_{j=1}^n [\sum_{i=0}^{j-1} \mu(i)]}$$

which is obviously just the $\exp_q(x)$ that appeared in section 4 when $\mu(i) = [i+1] - [i]$. For $\mu(i) = 1$, this is the usual exponential function and then (5.2) recovers the usual coherent states in quantum mechanics.

As we have known the expression of coherent states in the Fock representation and the transformation function (4.13) from Fock space to coordinate space, we can easily obtain its expression in coordinate representation, i.e. wavefunctions of eigenstates of a_q

$$\phi_\alpha(x) = \sum_{n=0}^{\infty}\langle x|n\rangle\langle n|\alpha\rangle = [\widetilde{\exp}_\mu(-|\alpha|^2)]^{1/2}\,\widetilde{\exp}_\mu\left[\alpha\left(x - \frac{\mathrm{d}_q}{\mathrm{d}_q x}\right)\right]\exp_{q^2}(-x^2/[2])\,. \tag{5.3}$$

The probability distribution of a deformed coherent state in Fock representation is

$$|\langle n|\alpha\rangle|^2 = \widetilde{\exp}_\mu(-|\alpha|^2)\frac{(|\alpha|^2)^n}{\prod_{j=1}^{n-1}[\sum_{i=0}^{j}\mu(i)]}\,. \tag{5.4}$$

Which is a deformation of Poisson distribution. The deformed coherent states are also not orthogonal to each other due to

$$\langle\beta|\alpha\rangle = [\widetilde{\exp}_\mu(-|\alpha|^2)\,\widetilde{\exp}_\mu(-|\beta|^2)]^{1/2}\,\widetilde{\exp}_\mu(\alpha\beta^*)\,. \tag{5.5}$$

The completeness relation for the deformed coherent states is shown to hold only in the case $\mu(x) = [x+1] - [x]$, i.e. q-coherent states (see appendix).

$$\int|\alpha\rangle\langle\alpha|\frac{\mathrm{d}_q^2\alpha}{\pi} = 1 \qquad \text{for} \quad \mu(x) = [x+1] - [x]\,. \tag{5.6}$$

This is an interesting consequence. In this case, (4.7) becomes

$$N_q|n\rangle = N_q|n\rangle \qquad a_q^+|n\rangle = ([n+1])^{1/2}|n\rangle \qquad a_q|\,n\rangle = ([n])^{1/2}|n\rangle\,. \tag{5.7}$$

On the basis of the completeness relation (5.6), we can expand an n-quantum state in terms of q-coherent states

$$|n\rangle = \int|\alpha\rangle\langle\alpha|n\rangle\frac{\mathrm{d}_q^2\alpha}{\pi} = \int[\exp_q(-|\alpha|^2)]^{1/2}\frac{\overline{\alpha}^n}{([n]!)^{1/2}}\frac{\mathrm{d}_q^2\alpha}{\pi}\bigg|\alpha\bigg\rangle \tag{5.8}$$

where $\overline{\alpha}$ stands for α^*. Substituting (5.9) into (5.8), we obtain

$$a_q^+\overline{\alpha}^n = \overline{\alpha}^{n+1} \qquad a_q\overline{\alpha}^n = [n]\overline{\alpha}^{n-1}\,. \tag{5.9}$$

Then we immediately have an expression of creation and annihilation operator in coherent states representation, i.e.

$$a_q^+ = \overline{\alpha} \qquad a_q = \frac{\mathrm{d}_q}{\mathrm{d}_q\overline{\alpha}}\,. \tag{5.10}$$

Any state of a harmonic oscillator must possess the following expansion in q-Forh space:

$$|\psi\rangle = \sum_{n=0}^{\infty}c_n|n\rangle \tag{5.11}$$

where $\sum |c_n|^2 = 1$. In order to expand the arbitrary state in terms of q-coherent states, we must use the completeness relation which has been used in deriving (5.9). Substituting (5.8) into (5.11), we obtain the following expansion in q-coherent representation:

$$|\psi\rangle = \int \sum_{n=0}^{\infty} c_n \frac{\bar{\alpha}^n}{([n])^{1/2}} [\exp_q(-|\alpha|^2)]^{1/2} \frac{d_q^2 \alpha}{\pi} \bigg| \alpha \bigg\rangle . \tag{5.12}$$

Obviously the amplitude distribution function in this representation is not an entire function

$$\langle \alpha | \psi \rangle = \chi(\bar{\alpha})[\exp_q(-\bar{\alpha}\alpha)]^{1/2} \tag{5.13}$$

where $\chi(\bar{\alpha})$ is an (anti-) analytical function on the complex α-plane and is defined by the expansion coefficients $\{c_n\}$ of the state $|\psi\rangle$ in Fock space, i.e.

$$\chi(\beta) = \sum_{n=0}^{\infty} c_n \frac{\beta^n}{([n]!)^{1/2}} . \tag{5.14}$$

There is apparently a one-to-one correspondence between the entire function (5.15) and the state in Fock space (5.12). The Hilbert space of such functions $\chi(\beta)$ is the known Bargman space [24], in which the inner product of two vectors φ and χ is defined by

$$\langle \varphi | \chi \rangle = \int [\varphi(\bar{\alpha})]^* \chi(\bar{\alpha}) \exp_q(-|\alpha|^2) \frac{d_q^2 \alpha}{\pi} . \tag{5.15}$$

This definition can be easily derived via q-coherent state representation, i.e. by using (5.12) and (5.14).

6. Conclusion and discussion

Above we have attempted to establish a deformation of quantum mechanics. In fact we considered a discrete quantum mechanics in one dimension, in which the intervals are not uniform. Instead, the intervals are divided by a geometric sequence. The Hermitian conjugation of the q-differential operator (strictly speaking quotient of q-difference) are discussion and then a one-dimensional positive-definite stationary Schrödinger equation is set up.

For the case of the Harmonic oscillator, we have solved the energy spectrum and the eigenfunctions by means of the operator method. Owing to the constraints of Jacobi identity, the oscillator algebra may contain an arbitrary function of the q-quantum number operator N_q only. In order to recover the usual quantum mechanics, this function is only equal to unity when the deformation parameter goes to unity. So the eigenvalues and eigenfunctions of the Hamiltonian contain a deformation function, which can be chosen according to experimental results.

Furthermore, we discussed the coherent states for the deformed Heisenberg–Weyl algebra. Certainly the deformed coherent states also contain the deformation function. However the coherent states satisfy the completeness relation only for a special deformation function. This is just the case of the known q-analogue of the Heisenberg–Weyl algebra, a Hopf algebra. Other potential cases and the three-dimensional case are now in discussion. Aside from the non-commutative geometry approach to deformations of quantum mechanics [25], it is also worthwhile to notice the connections between quantum groups and discrete quantum mechanics.

Appendix

From the definition (2.4), one can easily find

$$\frac{\mathrm{d}_q}{\mathrm{d}_q x} x^n = [n] x^{n-1} \tag{A1}$$

$$\frac{\mathrm{d}_q}{\mathrm{d}_q x} \exp_q x = \exp_q x \tag{A2}$$

$$\hat{q} x^n = q^n x^n \tag{A3}$$

$$[n]_{q^m} = [mn]/[m] \tag{A4}$$

$$\frac{\mathrm{d}_q}{\mathrm{d}_q x} f(x^m) = [m] x^{m-1} \frac{\mathrm{d}_{q^m}}{\mathrm{d}_{q^m}(x^m)} f(x^m)\,. \tag{A5}$$

The following formula of integration by parts is a direct consequence of (2.5):

$$\int_{x_i}^{x_f} (\hat{q} f)\, \mathrm{d}_q g = f g|_{x_i}^{x_f} - \int_{x_i}^{x_f} (\hat{q}^{-1} g)\, \mathrm{d}_q f\,. \tag{A6}$$

The q-analogue of the Γ-function is defined by

$$\Gamma_q(\rho) := \int_0^\infty x^{\rho-1} \exp_q(-x)\, \mathrm{d}_q x \overset{\text{or}}{=} \frac{[2]}{2} \int_{-\infty}^{+\infty} x^{2\rho-1} \exp_q(-x^2)\, \mathrm{d}_q x\,. \tag{A7}$$

Using (A1)–(A4), one can show that

$$\Gamma_q(\rho+1) = [\rho]\Gamma_q(\rho) \qquad \Gamma_q(n+1) = [n]!\,. \tag{A8}$$

The completeness relation (5.7) is shown in the following:

$$\begin{aligned}\int |\alpha\rangle\langle\alpha| \mathrm{d}_q^2\alpha &= \sum_{m,n} \frac{|\alpha\rangle\langle\alpha|}{([n]![m]!)^{1/2}} \int_0^\infty \mathrm{d}_q|\alpha| \exp_q(-|\alpha|^2)|\alpha|^{n+m+1} \int_0^{2\pi} \mathrm{d}\phi\, \mathrm{e}^{\mathrm{i}(n-m)\phi} \\ &= \pi \sum_n |n\rangle\langle n| = \pi\,. \end{aligned} \tag{A9}$$

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